# Module Categories:

Consider the category $\Delta^{+}$ whose:

- objects are $[n]$ and $[n]^+ := \{0<1<\dots<n<+\}$. $n\geq 0$.
- morphisms: (i) $[n]\to[m]$ functors.

  (ii) $[n]\to[m]^+$ essential image does not contain $+$.

  (iii) $f:[n]^+\to[m]^+$ send $+$ to $+$, (and only)

**Def'n:** Given a st. monoidal $\infty$-category $A$ a st. $\infty$-cat. $\mathcal{C}$ is an *A-module* if one has a functor:

$$(A,\mathcal{C})^{\otimes}:\Delta^{+,op}\to Cat_\infty^{st.} \quad \text{s.t.}$$

- $(A,\mathcal{C})^{\otimes}|_{\Delta^{op}}:\Delta^{op}\to Cat_\infty^{st.}$ is the $\otimes$-str. of $A$.
- $\forall\, n\geq 0$

$$(A,\mathcal{C})^{\otimes}([n]^+)\xrightarrow{\simeq}(A,\mathcal{C})^{\otimes}([n])\times(A,\mathcal{C})^{\otimes}([0]^+).$$

$[n]\to[n^+]$. & $[0]^+\to[n]^+$

$i\mapsto i$ $\qquad 0\mapsto n$, $+\mapsto +$.

- $(A,\mathcal{C})^{\otimes}([0]^+)=\mathcal{C}$.

Consequences: - $(A,\mathcal{C})^{\otimes}([1]^+)\simeq A\times\mathcal{C}$.

- the morphism $[0]^+\to[1]^+$ gives

$$a_{\mathcal{C}}: A\times\mathcal{C}=(A,\mathcal{C})^{\otimes}([1]^+)\to(A,\mathcal{C})^{\otimes}([0]^+)=\mathcal{C}.$$

the action of $A$ on $\mathcal{C}$. $a_{\mathcal{C}}(a,x):=a\otimes x$.

All the rest encodes compatibilities. (coherence data).

Example: (i) let $j : \Delta^{+} \to \Delta$
$$[n] \mapsto [n], \quad [n]^{+} \mapsto [n+1].$$
then if $A^{\otimes} : \Delta^{op} \to \mathrm{Cat}_{\infty}^{st.}$ is a monoidal $\infty$-cat. then

(Check this!). $A^{\otimes} \circ j : \Delta^{+,op} \to \mathrm{Cat}_{\infty}^{st.}$ makes $A$ into an $A$-module.

(ii) In particular, Vect is a module over itself and so is Spectr.

Given $A^{\otimes}$, we let $A\text{-mod} := \mathrm{Cat}_{\infty}^{\mathrm{Mon}^{+}} \times_{\mathrm{Cat}_{\infty}^{\mathrm{Mon}}} \{A^{\otimes}\}$,

where $\mathrm{Cat}_{\infty}^{\mathrm{Mon}} \subseteq \mathrm{Fun}(\Delta^{op}, \mathrm{Cat}_{\infty})$ is the subcategory spanned by monoidal fctors. &
$\mathrm{Cat}_{\infty}^{\mathrm{Mon}^{+}} \subseteq \mathrm{Fun}(\Delta^{op,+}, \mathrm{Cat}_{\infty})$ ———
— module category structures.

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Def'n: Given monoidal categories $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}$ a non-unital right-lax monoidal functor is a map:
$$F_{\mathcal{D}} : \mathcal{C}^{\otimes,\Delta^{op}} \to \mathcal{D}^{\otimes,\Delta^{op}}$$
of coCart. fibrations $/\Delta^{op}$ s.t. any inert morphism of $\mathcal{C}^{\otimes,\Delta^{op}}$ maps to an inert morph. in $\mathcal{D}^{\otimes,\Delta^{op}}$

$\mathcal{C}^{\otimes,\Delta^{op}} \xrightarrow{p} \Delta^{op}$ a morph. $\alpha$ in $\mathcal{C}^{\otimes,\Delta^{op}}$ is inert if it is coCart. & $p(\alpha)$ is inert, i.e. $p(\alpha) : [n] \to [m]$ s.t. $\forall j \in [m]^{\circ}$ $|p(\alpha)^{-1}(j)| = 1$.

Exercise: The above cond. is equivalent to requiring that F sends coCartesian morphisms $\alpha$ in $\mathcal{C}^{\otimes, \Delta^{op}}$ whose image $p(\alpha)$ is a <u>spine morphism</u> to coCart.

In $\Delta$ for $n \geq 1$, let $\rho_i : [1] \to [n]$
$(0 \to 1) \mapsto (i \to i+1)$.

$\mathrm{sp}_n : [1]^{\amalg n} \to [n]$. is given by $\coprod_{1 \leq i \leq n} \rho_i$.

The cond. is $p(\alpha) = \mathrm{sp}_n$ for some $n \geq 1$.

A non-unital assoc. algebra is $F: *^{\Delta^{op}} \to \mathcal{C}^{\otimes, \Delta^{op}}$ a non-unital right-lax monoidal functor.

Similarly, given $(\mathcal{A}, \mathcal{C})^{\otimes}$ & $(\mathcal{A}', \mathcal{C}')^{\otimes}$ two module categories. a non-unital right-lax functor between them is
$F: (\mathcal{A}, \mathcal{C})^{\otimes, \Delta^{+,op}} \to (\mathcal{A}', \mathcal{C}')^{\otimes, \Delta^{+,op}}$ that sends.
morphisms over: $\mathrm{sp}_n : [n] \to [1]^{*n}$ in $\Delta^{op} (\subseteq \Delta^{+,op})$ &
$\mathrm{act}_n : [n]^+ \to [n] \times [0]^+$ in $\Delta^{+,op}$.
to coCartesian morphisms.

<u>Def'n:</u> let $(\mathcal{A}, \mathcal{C})^{\otimes}$ be a module category the category:
$\mathrm{AssocAlg} + \mathrm{mod}(\mathcal{A}, \mathcal{C})$ is that of (non-unital) right-lax functors:

$*^{\Delta^{+,op}} \to (\mathcal{A}, \mathcal{C})^{\otimes, \Delta^{+,op}}$.

Restriction to $\Delta^+$ gives a map $\mathrm{Assoc.Alg} + \mathrm{mod}(\mathcal{A}, \mathcal{C}) \to \mathrm{Assoc.Alg}(\mathcal{A})$.

For an object $A \in \mathrm{Assoc.Alg}(\mathcal{A})$ let $A\text{-mod}(\mathcal{C})$ be the fiber of.

<u>Example:</u> (i) $\mathrm{DGCat} := \mathrm{Vect}\text{-mod}(\mathrm{Cat}_{\infty}^{st.})$.

One last example of (symmetric) monoidal $\infty$-category is the following.

Let $\underline{C}$ be a sym. mon. model structure, i.e. $1_{\underline{C}}$ is cofibrant, $\underline{C}$ is closed & $\otimes : \underline{C} \otimes \underline{C} \to \underline{C}$ is a left Quillen fctor.

Consider: $N(\underline{C}_c)[W^{-1}]$. $\underline{C}_c \subseteq \underline{C}$ subcategory of cofibrant objects.

[HA, Prop. 4.1.7.4].

$W$ = weak equivalences.

The operation above is given $(\mathcal{C}, W)$ an $\infty$-cat. w/ $W \subseteq \mathrm{Fun}([1], \mathcal{C})$ s.t. $W$ contains isomorphisms & is stable under homotopies & compositions. $\exists$ $\mathcal{C}[W^{-1}]$ an $\infty$-category initial among those sending morphisms in $W$ to isomorphism. Concretely, this is obtained by taking a fibrant replacement of $(\mathcal{C}, W)$ in marked simplicial sets.

<u>Sub-example:</u> given $\underline{C}$ a model category (not necessarily a simplicial model category) then $N(\underline{C})[W^{-1}]$ is the $\infty$-cat. underlying $\underline{C}$. In particular, it does not depend on the fibrations & cofibrations only depends on the weak equivalences.

<u>Prop:</u> [4.1.7.4]. - The $\infty$-category $N(\underline{C}_c)[W^{-1}]$ has a $\otimes$-str.

[4.1.7.6]. - If $\underline{C}$ is a simplicial sym. mon. model category, then $N(\underline{C}_{cf}) \simeq N(\underline{C}_c)[W^{-1}]$ as sym. mon. $\infty$-cats. and $N(\underline{C}_c)^{\otimes} \to N(\underline{C}_c)[W^{-1}]$ is sym. monoidal.

[HA, 7.1.2.8/ 7.1.2.11/ 7.1.2.12]. <u>Cor:</u> For $k$ a comm. ring. let $\underline{Ch}(k)$ be the model structure. where: w.e. are q.-iso., fibrations are levelwise surjective maps, cofib. defined by L.L.P. Then $\underline{Ch}(k)$ w/ tensor product of chain cplxes. is a sym. monoidal model category.

In particular, $N(\underline{Ch}(k)_c)[W^{-1}] \simeq \mathcal{D}(k)$ as $\otimes$-$\infty$-cats. w/

$N(\underline{Ch}(k)_c) \to \mathcal{D}(k)$ a $\otimes$-monoidal functor.